DATE OF EXAM: September 8, 2015 **Solution**

SUBJECT NAME: Probability **MIDTERM Exam** **2015-16** **Semester I**

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1. Suppose we toss a fair coin two times. Let X denote the number of heads on the first toss, Y denote the number of tails on the first toss and Z denote the number of heads on the second toss.

   (a) Write out the Sample space and define the Probability function for the above experiment.
   Solution. Sample space $S = \{HH, HT, TH, TT\}$.
   The possible values of X are $0, 1$. The corresponding probabilities are $1/2, 1/2$.
   The possible values of Y are $0, 1$. The corresponding probabilities are $1/2, 1/2$.
   The possible values of Z are $0, 1$. The corresponding probabilities are $1/2, 1/2$.
   Let $E_i \ \ (i = 1, 2, 3, 4)$ be an event of the sample space, then $P(E_i) = 1/4$.

   (b) Show that $X, Y, Z \sim Bernoulli(1/2)$.
   Solution. We have $P(X = 0) = P(\text{No head in first toss}) = P(\text{Tail in first toss}) = 1/2$ and also, $P(X = 1) = 1/2$. Therefore, $X \sim Bernoulli(1/2)$.
   Similarly, $Y \sim Bernoulli(1/2)$ and $Z \sim Bernoulli(1/2)$.

   (c) Let $W = X + Y$. Find the distribution of W.
   Solution. If A and B are two independent random variables and $A \sim Bernoulli(p)$, $B \sim Bernoulli(p)$, then $A + B \sim Binomial(2, p)$.
   Thus, $W = X + Y \sim Binomial(2, 1/2)$ and the distribution of W is given by:

$$\begin{aligned} P(W = k) &= \binom{2}{k}\left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{2-k}, \quad (k = 0, 1, 2) \\ &= \binom{2}{k}\left(\frac{1}{2}\right)^2. \end{aligned}$$

   (d) Let $U = X + Z$. Find the distribution of U.
   Solution. $U = X + Z \sim Binomial(2, 1/2)$ and the distribution of U is given by:

$$P(U = k) = \binom{2}{k}\left(\frac{1}{2}\right)^2.$$

   (e) Do W and U have the same distribution? Explain your answer.
   Solution. Yes. Since X and Y are independent. Also, X and Z are independent.

2. Shyam is randomly selected from the citizens of Hyderabad by the Health authorities. A laboratory test on his blood sample tells Shyam that he has tested positive for H1N1 virus. It is found that 95% of people with H1N1 test positive but 2% of people without the disease will also test positive. Suppose that 1% of the population has the disease. What is the probability that Shyam does not have H1N1 disease ?

Solution. Let d denote the event where the population has the disease and T^+ be the event where the test is positive. So, $P(d) = 1\%$ and therefore, the probability that population does not has the disease is $P(d') = 1 - P(d) = 99\%$. Also, given: $P(T^+|d) = 95\%$ and $P(T^+|d') = 2\%$.
To find: $P(d'|T^+)$.
Using Baye's Theorem, we can write

$$\begin{aligned} P(d'|T^+) &= \frac{P(d') \cdot P(T^+|d')}{P(T^+|d') + P(T^+|d)} \\ &= \frac{99\% \cdot 2\%}{95\% + 2\%} \\ &= \frac{99\% \cdot 2\%}{97\%} \\ &= 2.041\% \end{aligned}$$

Thus, the probability that Shyam does not have H1N1 disease is 2.041%.

3. Let $0 < p, q < 1$. Let X be a Geometric (p) random variable and Y be an independent Geometric (q) random variable. Let $W = max\{X, Y\}$. Find the distribution of W.

Solution.

$$\begin{aligned} P(W = k) &= P(max(X, Y) = k) \\ &= P(max(X, Y) \leqslant k) - P(max(X, Y) \leqslant k - 1) \qquad (1) \end{aligned}$$

Now, $P(max(X, Y) \leqslant k) = P(X \leqslant k, Y \leqslant k) = P(X \leqslant k)\ P(Y \leqslant k)$, since X and Y are independent.
Since $X \sim Geom(p)$, we have

$$P(X \leqslant k) = \sum_{j=1}^{k} (1-p)^{j-1}\ p = (1 - (1-p)^k).$$

Similarly, we have $P(Y \leqslant k) = \sum_{j=1}^{k} (1-q)^{j-1}\ q = (1 - (1-q)^k)$.
Substituting in (1), we obtain:

$$\begin{aligned} P(W = k) &= P(max(X, Y) = k) \\ &= (1 - (1-p)^k)\ (1 - (1-q)^k) - (1 - (1-p)^{k-1})\ (1 - (1-q)^{k-1}). \end{aligned}$$

4. Let X and Y be discrete random variables with $Range(X) = \{0, 1, 2\}$ and $Range(Y) = \{1, 2\}$ with joint distribution given by the chart below.

	X=0	X=1	X=2
Y=1	0.1	0.2	0.1
Y=2	0.3	0.2	0.1

(a) What is the marginal probability mass function of Y ?

Solution. The marginal probability mass function of Y is given by

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

We have

$$f_Y(1) = 0.1 + 0.2 + 0.1 = 0.4 \text{ and } f_Y(2) = 0.3 + 0.2 + 0.1 = 0.6$$

$\therefore$

$$f_Y(y) = \begin{cases} 0.4 & \text{for } y = 1 \\ 0.6 & \text{for } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) What is the value of $P(X = 1|Y = 2)$?

Solution.

$$\begin{aligned} P(X = 1|Y = 2) &= \frac{P(X = 1, Y = 2)}{P(Y = 2)} \\ &= \frac{p_{XY}(1, 2)}{P(Y = 2)} \\ &= \frac{0.2}{0.3 + 0.2 + 0.1} = 0.333 \end{aligned}$$

(c) Find $E[XY]$.

Solution.

$$\begin{aligned} E[XY] &= (0 \times 1) \times 0.1 + (1 \times 1) \times 0.2 + (2 \times 1) \times 0.1 + (0 \times 2) \times 0.3 \\ &\quad +(1 \times 2) \times 0.2 + (2 \times 2) \times 0.1 \\ &= 1.3 \end{aligned}$$

(d) Find $Cov(X, Y) := E[XY] - E[X]\ E[Y]$.

Solution.

$$\begin{aligned} E[X] &= 0 \times 0.4 + 1 \times 0.4 + 2 \times 0.2 \\ &= 0.8 \end{aligned}$$

and

$$\begin{aligned} E[Y] &= 1 \times 0.4 + 2 \times 0.6 \\ &= 1.6 \end{aligned}$$

Thus, $Cov(X, Y) := E[XY] - E[X]\ E[Y] = 1.3 - (0.8)\,(1.6) = 0.02$

(e) Are X and Y independent?

Solution. No, since $P(X = 1, Y = 2) \neq P(X = 1)\ P(Y = 2)$.